METRIC FIXED POINT THEOREMS

FOR GENERALIZED CONTRACTIONS

SHORT SYNOPSIS

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Abstract

The short synopsis consists of **five sections** namely, Introduction, Literature review, Objectives of research, Methodology to be adopted and Proposed/Expected Outcome of Research.

The **first section** sets a brief introduction to the evolution and development of fixed point theory and presents some concerned basic fundamentals.

The **second section** is divided into **two subsections**. The **first subsection**, titled Generalizations of Contractive Conditions in Metric Spaces aims to highlight some extentions of Banach Contraction Principle within the framework of metric spaces. On the other hand, **the second subsection** titled Extensions of Fixed Point Results in Various Spaces focuses on several definitions and results related to extension of several contraction mappings in the setting of some more general spaces.

The **third section** deals with various objectives of the underlying research.

The **fourth section** seeks to explore the methodology involved in underlying research.

The **fifth section** discusses some proposed and expected outcomes of the underlying research.

Keywords: fixed point, banach contraction, kannan contraction, G-contraction, F-contraction.

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1. Introduction

Fixed point theory is an extremely rich and fascinating branch of mathematics that is open to many domains such as Functional Analysis, Differential Equations, Classical Analysis, Metric Space, Operator Theory, Topology, Discrete Mathematics and various others. It is one of the strongest and most popular instruments for modern mathematical analysis. A self-map $\mho$ defined on some non empty set A is said to have a fixed point say a if it remains invariant under $\mho$ that is, $\mho(a) = a$. Studying the presence of fixed points of a mapping is a very crucial property of a mapping. It is possible that a given mapping does not have any fixed point, or if it has, it may be unique fixed point or multiple. Fixed point theorems are those results, in which one explores various conditions on the underlying set and the mapping ensuring that fixed point(s) exist and sometimes also concerns properties of fixed points such as uniqueness. In fact, fixed point theory is that branch of mathematics that entails the study of those self-mappings which leaves at least one element invariant. Different kinds of fixed points exist in literature such as common fixed points, common coincidence points, common coupled fixed points, and many more.

In pure mathematics the focus lies in the development and improvement of conditions for proving the existence and uniqueness of solutions to different types of problems, but applied mathematics investigates the methods or techniques for obtaining these solutions.

The area of fixed point has its origin from a simple problem of solving an equation. In general, such a problem is transformed to an equivalent problem of evaluating the fixed points of some appropriate mapping defined on some suitable domain. There are often a variety of other mathematical problems like that of solving a given system of equations. Again, such a problem can usually be converted to the problem of exploring fixed points of some suitable mapping $\mho$ specified on some suitable set A. In analysis, we usually encounter equations of the form $\mho(a) = 0$ and $\mho(a) = a$ where the idea is to solve them. Different strategies are as of now accessible, some of these lead us to exact solutions while some of them provides an approximate solution to the problem. Fixed point results have additionally been broadly used to handle such problems. To understand this, a quadratic equation $\xi^2 - 4\xi + 3 = 0$ is considered whose roots are $\xi = 3$ and $\xi = 1$, it may be observed that this equation can be rewritten as follows

$$\frac{\xi^2+3}{4}=\xi.$$

So if a real valued function $\mho$ is defined by $\mho(\xi)=\frac{\xi^2+3}{4}$, then the quadratic equation reduces to the equation $\mho(\xi)=\xi$. It may be noticed that $\xi=3$ and $\xi=1$ are the two fixed points of $\mho$. Therefore it may be concluded from the above that finding solution of a functional equation $\mho(\xi)-\xi=0$, is equivalent to investigating for fixed point of some suitable mapping $\mho$.

Fixed point theory contributes as a strong theoretical tool in order to prove various existence and uniqueness theories as that of differential equations, integral equations and generally in proving different kinds of existence theorems for several types of equations. The scope of fixed point theory lies beyond mathematics as well in many research problems that may be reduced to fixed point problems. Metric fixed point theory is one of the most popular branches of fixed point theory where the underlying set is equipped with some metric conditions. There exists a massive literature on this subject and still this continues to be a very active and fertile field of analysis.

Formally fixed point theory began in the initial twentieth century as an important area of analysis. The general thought of this principal hypothesis established in 1922 was the extraordinary work of a polished mathematician S. Banach [1] which proposes an appropriate condition on the metric space under which the existence and uniqueness of a fixed point is guratneed. Also it is interesting to note that the proof contains a constructive method to find that fixed point. It stands as the most fundamental and one of the most versatile results in the area of metric fixed point theory. This classical result is named as Banach Fixed Point Theorem or the Banach Contraction Principle (BCP). It states that "every contraction mapping on any complete metric space into itself possesses a unique fixed point". This result is formally credited as the foundation to metric fixed point theory.

2. Literature Review

Picard, 1890 [48] introduced Picard's Convergence Theorem and proved the existence of solutions to nonlinear equations of the form $\mho(\check{\xi})=\check{\xi}$ by introducing an iterative sequence, called *Picard iterative sequence* given by $\check{\xi}_{n+1}=\mho(\check{\xi}_n)$.

Theorem 1. *(Picard [48]): Let* $\mho$ *be a real valued function which is assumed to be continuous on [p, q] and be differentiable on (p, q). Suppose for some number L with* $0 \leq L < 1$ *the following is true :*

$$|\mho'(\check{\xi})| \leq L \quad \text{for each } \check{\xi} \in (p, q).$$

Then starting with any $\check{\xi}_0 \in (p, q)$ *the sequence* $\check{\xi}_n$ *of points in* (p, q) *defined by* $\check{\xi}_{n+1} = \mho\check{\xi}_n$ *for* $n \geq 0$ *will converge to some fixed point* $\check{z}$ *of* $\mho$*, that is, it satisfies the equation* $\mho\check{z} = \check{z}$.

Banach, 1922 [1] introduced a contraction mapping and presented the most fundamental theorem concerning fixed points well known as Banach's Contraction Principle. This result states that "every contraction mapping on a complete metric space will always possess a fixed point and such a fixed point is unique". The proof of the theorem involves Picard iteration to approach the fixed point.

Theorem 2. *(Banach [1]): Let* $(\mathcal{Z}, \zeta)$ *be a metric space and suppose* $\mho: \mathcal{Z} \to \mathcal{Z}$ *is a contraction mapping that is, there exists some number L,* $0 \leq L < 1$ *satisfying the inequality*

$$\zeta(\mho\check{\xi}, \mho\check{\varrho}) \leq L\zeta(\check{\xi}, \check{\varrho}) \quad \text{for each } \check{\xi}, \check{\varrho} \in \mathcal{Z}.$$

If the metric space $\mathcal{Z}$ *is assumed to be complete then the following is true:*

1. $\mho$ *possesses exactly one fixed point say* $z \in \mathcal{Z}$.

2. *Further, for every* $\check{\xi}_0 \in \mathcal{Z}$*, the sequence* $\check{\xi}_n$ *defined by* $\check{\xi}_{n+1} = \mho\check{\xi}_n \; ; n \geq 0$ *will always converge to the above fixed point z.*

He further proved that a contraction mapping is always continuous.

These celebrated results have further witnessed a variety of extensions and generalizations in several directions by many mathematicians and researchers for both single valued as well as for multivalued mappings. Fixed point and related results have been explored in literature under several combinations of contractive conditions and different generalizations of metric spaces.

a. Generalizations of Contractive Conditions in Metric Spaces

This sections mentions some of the extensions of BCP in a metric space. The metric space $(\mathcal{Z}, \zeta)$ considered in the subsequesnt theorems of this section is assumed to be complete. One of the earliest extensions of BCP was given by Boyd and Wong, 1969 [16]. The authors replaced the contraction constant by a special control function thereby obtaining a more general result of fixed points of such mappings given below.

Theorem 3. *(Boyd and Wong [16]): Let* $\Phi: [0, \infty) \to [0, \infty)$ *is an upper semi-continuous (u.s.c) function from the right satisfying* $0 \leq \Phi(a) < a$ *for all* $a > 0$.

If $\mho: \mathcal{Z} \to \mathcal{Z}$ *satisfies*

$$\zeta(\mho\check{\xi}, \mho\check{\varrho}) \leq \Phi(\zeta(\check{\xi}, \check{\varrho}))$$

for all $\check{\xi}, \check{\varrho} \in \mathcal{Z}$, *then it has a unique fixed point* $\check{z} \in \mathcal{Z}$ *with* $\{\mho^n\check{\xi}\}$ *converging to* $\check{z}$ *for all* $\check{\xi} \in \mathcal{Z}$.

Subsequently Ciric, 1974 [6] presented a weaker class of contraction mapping called quasi-contractive mapping and proved the existence of a fixed point for such mappings on a complete metric space.

Theorem 4. *(Ciric [6]): If* $\mho: \mathcal{Z} \to \mathcal{Z}$ *satisfies the following for some* $0 \leq c < 1$,

$$\zeta(\mho\check{\xi}, \mho\check{\varrho}) \leq c \cdot max\{\zeta(\check{\xi}, \check{\varrho}), \zeta(\check{\xi}, \mho\check{\xi}), \zeta(\check{\varrho}, \mho\check{\varrho}), \zeta(\check{\xi}, \mho\check{\varrho}), \zeta(\check{\varrho}, \mho\check{\xi})\}$$

for all $\check{\xi}, \check{\varrho} \in \mathcal{Z}$. *Then, a unique fixed point of* $\mho$ *exists in* $\mathcal{Z}$.

Caristi, 1976 [15] generalized the Banach contraction mapping in a different direction and presented the concept of a Caristi's contraction and further explored fixed points of such mappings.

Theorem 5. *(Caristi [15]): Let* $\mho: \mathcal{Z} \to \mathcal{Z}$ *be a mapping satisfying*

$$\zeta(\check{\xi}, \mho\check{\xi}) \leq \phi(\check{\xi}) - \phi(\mho\check{\xi})$$

for all $\check{\xi} \in \mathcal{Z}$. *If the function* $\phi: \mathcal{Z} \to [0, \infty)$ *is assumed to be a lower semi-continuous function, then* $\mho$ *has a fixed point in* $\mathcal{Z}$.

Note that a Banach contraction is a special case of Caristi's contraction if ϕ is set as

$$\phi(\check{\xi}) = \frac{\zeta(\check{\xi}, \mho\check{\xi})}{1 - L}$$

for some $0 \leq L < 1$.

As proved by Banach that every banach contraction is continuous, it is quite natural to contemplate the following question:Can we find any such contractive conditions on a discontinuous mapping also ? In other words, under what sufficient conditions can a discontinuous function also have fixed points ? The answer for this is in affirmative and this was another important development in th area of fixed point theory. Kannan, 1968 [4] introduced an entirely new operator satisfying a different condition and proved Kannan's fixed point theorem for such a contractive mapping. This result is well known as Kannan's contraction principle.

Theorem 6. *(Kannan [4]): Let* $\mho: \mathcal{Z} \to \mathcal{Z}$ *be a mapping and assume there exists some number M,* $0 \leq M \leq \frac{1}{2}$ *such that*

$$\zeta(\mho\check{\xi}, \mho\check{\varrho}) \leq M[\zeta(\mho\check{\xi}, \check{\xi}) + \zeta(\mho\check{\varrho}, \check{\varrho})]$$

for all $\check{\xi}, \check{\varrho} \in \mathcal{Z}$. *Then,* $\mho$ *has exactly one fixed point in* $\mathcal{Z}$.

Reich [17], [18], Ciric [6] and Rus [19] attempted to generalize both Banach and Kannan type contractions by composing the Banach and Kannan maps and defined Ciric-Reich-Rus operator $\mho$ such that $\mho: \mathcal{Z} \to \mathcal{Z}$ satisfies the condition

$$\zeta(\mho\check{\xi}, \mho\check{\varrho}) \leq f\zeta(\check{\xi}, \check{\varrho}) + g\zeta(\check{\xi}, \mho\check{\xi}) + h\zeta(\check{\varrho}, \mho\check{\varrho})$$

for all $\check{\xi}, \check{\varrho} \in \mathcal{Z}$ where $f + g + h < 1$. It is proved that in a complete metric space $(\mathcal{Z}, \zeta)$, every Ciric-Reich-Rus operator possesses exactly one fixed point.

Chatterjea S.K., 1972 [5] investigated the existence of fixed points for a mapping imposed with a similar contraction condition

$$\zeta(\mho\check{\xi}, \mho\check{\varrho}) \leq L[\zeta(\mho\check{\xi}, \check{\varrho}) + \zeta(\mho\check{\varrho}, \check{\xi})]$$

for all $\check{\xi}, \check{\varrho} \in \mathcal{Z}$ and some $0 \leq L \leq \frac{1}{2}$ as that of Kannan condition.

Further, Wardowski, 2012 [21] initiated a new concept of an ϝ-contraction mapping which involves a special category of a real-valued function ϝ with domain as the set of positive real numbers. This function further possesses some supplementary properties. His study involved the investigation of fixed points of mapping which satisfied the condition of existence of some $\Upsilon > 0$ such that

$$\Upsilon + \text{ϝ}[\zeta(\mho\check{\xi}, \mho\check{\varrho})] \leq \text{ϝ}[\zeta(\check{\xi}, \check{\varrho})]$$

for all $\check{\xi}, \check{\varrho} \in \mathcal{Z}$ such that $\mho\check{\xi} \neq \mho\check{\varrho}$.

Later, Batra and Vashishtha, 2014 [22] extended and generalized the concepts of a G-contraction [20] and an ϝ-contraction to form a bigger class called ϝ-G-contractions. This class carried features of both ϝ-contraction as well as of G-contraction. The authors considered the following conditions on the mapping $\mho$:

(a) $\mho$ is edge preserving, that is :

$$(\check{\xi}, \check{\varrho}) \in E_G \Rightarrow (\mho\check{\xi}, \mho\check{\varrho}) \in E_G.$$

(b) For some $\Upsilon > 0$ the following holds:

$$\Upsilon + \text{ϝ}[\zeta(\mho\check{\xi}, \mho\check{\varrho})] \leq \text{ϝ}[\zeta(\check{\xi}, \check{\varrho})]$$

for all $(\check{\xi}, \check{\varrho}) \in E_G$ such that $\mho\check{\xi} \neq \mho\check{\varrho}$.

Alber and Delabriere, G., 1997 [38] induced the concept of the weak contraction which was defined for single-valued maps and fixed point existence results have been proved for such weak contractions by many mathematicians on various general spaces (Rhoades, 2001 [39]). Mappings studied by them were defined with the support of a special control function ϕ, known as ϕ-weak contraction (or weakly contractive) and satisfied the contraction condition $\zeta(\mho\check{\xi}, \mho\check{\varrho}) \leq \zeta(\check{\xi}, \check{\varrho}) - \phi(\zeta(\check{\xi}, \check{\varrho}))$ for all $\check{\xi}, \check{\varrho} \in \mathcal{Z}$. Under certain control conditions, fixed points of such mappings were proved to exist.

Dutta and Choudhury, 2008 [41] generalized these weakly contractive maps by considering two control functions ϕ and ψ and defined a contraction condition on $\mho$ as $\psi(\zeta(\mho\check{\xi}, \mho\check{\varrho})) \leq \psi(\zeta(\check{\xi}, \check{\varrho})) - \phi(\zeta(\check{\xi}, \check{\varrho}))$ for all $\check{\xi}, \check{\varrho} \in \mathcal{Z}$ and investigated these operators for fixed points. These

maps were further generalized by Doric, 2009 [42] by replacing the metric function ζ by a more general function.

Ovidiu Popescu, 2011 [40] proved that some of the control conditions as considered by Doric could be milder thereby extending results to a larger class of mappings. Contraction conditions were weakened by several authors to produce new types of weak contractions and a variety of generalizations of such new weak contractions exist in literature. One may refer to [26]-[29] and some references there in.

b. Extensions of Fixed Point Results in Various Spaces

In literature, there are many generalizations of metric spaces and Banach Fixed point Theorem. In this sequel, Bakhtin, 1989 [30] presented a more general metric space called b-metric space and proved an analogue of Banach Contraction Principle in the framework of this space. Subsequently in 1993, Czerwik [43] made an attempt to extend some of the fixed point results with underlying space as a b-metric space. With this newly developed idea, several researchers introduced varied generalizations of the BCP in b-metric space (see e.g. [43]-[47])

Definition 1. *(Bakhtin [30]): Let $\mathcal{Z}$ be a non empty set. A function $b: \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^+$ is defined to be a "b-metric on $\mathcal{Z}$" if the conditions specified below hold :*

1. *$b(\check{\xi}, \check{\varrho})=0$ if and only if $\check{\xi} = \check{\varrho}$ (identity),*
2. *$b(\check{\xi}, \check{\varrho}) = b(\check{\varrho}, \check{\xi})$ (symmetry),*
3. *$b(\check{\xi}, \check{\varrho}) \leq s[b(\check{\xi}, z) + b(z,\check{\varrho})]$ for all $\check{\xi}, \check{\varrho}, z \in \mathcal{Z}$ for some $s \geq 1$ (triangle inequality).*

The pair $(\mathcal{Z}, b)$ is called a "b-metric space."

On the other hand, Branciari, A., 2000 [31] introduced a more general space called rectangular metric space. He generalized the triangle inequality by replacing it with a more general inequality called rectangular inequality. Banach Contraction theorem was also proved and thus extended in the setting of this space.

Definition 2. *(Branciari [31]): Consider a non empty set $\mathcal{Z}$. A function $: \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^+$ is defined as a "rectangular metric on $\mathcal{Z}$" if the conditions mentioned below are true :*

1. $\zeta(\check{\xi}, \check{\varrho})=0$ *if and only if* $\check{\xi} = \check{\varrho}$ *(identity),*

2. $\zeta(\check{\xi}, \check{\varrho}) = \zeta(\check{\varrho}, \check{\xi})$ *(symmetry),*

3. $\zeta(\check{\xi}, \check{\varrho}) \leq \zeta(\check{\xi}, z)+\zeta(z,u)+\zeta(u, \check{\varrho})$ *for all* $\check{\xi}, \check{\varrho}, z, u \in \mathcal{Z}$*. (rectangular inequality).*

The pair $(\mathcal{Z}, \zeta)$ *is called a "rectangular metric space."*

Matthews, 1994 [34] established another important generalization of a metric space. Motivated by data flow network, the author introduced the notion of partial metric as a part of his study. An interesting property of such a type of metric is that the self-distance of any point of the space may be not be always zero. The condition $\zeta(\check{\xi}, \check{\xi}) = 0$ was replaced by $\zeta(\check{\xi}, \check{\xi}) \leq \zeta(\check{\xi}, \check{\varrho})$ for all $\check{\xi}, \check{\varrho}$ in the definition of a metric. Further, the author proved the validity of Banach contraction principle in partial metric space. Since then, numerous authors have focused on this objective of generalization of various fixed point results from metric spaces to this larger class of partial metric spaces.

Definition 3. *(Matthews [34]): Consider a non empty set* $\mathcal{Z}$*. A function* $p: \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^+$ *is defined to be a "partial metric on* $\mathcal{Z}$*" if the conditions specified below hold :*

1. $p(\check{\xi}, \check{\xi}) = p(\check{\xi}, \check{\varrho}) = p(\check{\varrho}, \check{\varrho})$ *implies* $\check{\xi} = \check{\varrho}$ *(identity),*

2. $p(\check{\xi}, \check{\xi}) \leq p(\check{\xi}, \check{\varrho})$ *(small self-distance),*

3. $p(\check{\xi}, \check{\varrho}) = p(\check{\varrho}, \check{\xi})$ *(symmetry),*

4. $p(\check{\xi}, \check{\varrho}) \leq p(\check{\xi}, z) + p(z, \check{\varrho}) - p(z, z)$ *for all* $\check{\xi}, \check{\varrho}, z \in \mathcal{Z}$ *(triangle inequality).*

The pair $(\mathcal{Z}, p)$ *is called a "partial metric space."*

Satish Shukla, 2014 [35] hybridized partial metric and b-metric to produce partial b-metric. Analogous results of Banach contraction principle and Kannan type fixed point result were proved in by the author in his work in the framework of such space.

Definition 4. *(Satish Shukla [35]): Consider a non empty set* $\mathcal{Z}$*. A function* $pb: \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^+$ *is defined to be a "partial b-metric on* $\mathcal{Z}$*" if the conditions specified below hold :*

1. *$pb(\check{\xi}, \check{\xi}) = p(\check{\xi}, \check{\varrho}) = pb(\check{\varrho}, \check{\varrho})$ implies $\check{\xi} = \check{\varrho}$ (identity),*

2. *$pb(\check{\xi}, \check{\xi}) \leq pb(\check{\xi}, \check{\varrho})$ (small self-distance),*

3. *$pb(\check{\xi}, \check{\varrho}) = pb(\check{\varrho}, \check{\xi})$ (symmetry),*

4. *$pb(\check{\xi}, \check{\varrho}) \leq s[pb(\check{\xi}, z) + pb(z, \check{\varrho})] - pb(z, z)$ for all $\check{\xi}, \check{\varrho}, z \in \mathcal{Z}$ for some $s \geq 1$ (triangle inequality).*

The pair $(\mathcal{Z}, pb)$ is called a "partial b-metric space."

George, R. *et. al*. 2015 [32] gave the concept of rectangular b-metric space by blending rectangular metric and b-metric space properties. Various scholars have further generalized this space and proved results concerning fixed points of different contraction mappings.

Definition 5. *(George, R. et. al [32]): Let $\mathcal{Z}$ be a non empty set. A function $\zeta b : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^+$ is defined to be a "rectangular b-metric on $\mathcal{Z}$" if the conditions specified below hold :*

1. *$\zeta b(\check{\xi}, \check{\varrho}) = 0$ if and only if $\check{\xi} = \check{\varrho}$ (identity),*

2. *$\zeta b(\check{\xi}, \check{\varrho}) = \zeta b(\check{\varrho}, \check{\xi})$ (symmetry),*

3. *$\zeta b(\check{\xi}, \check{\varrho}) \leq s[\zeta b(\check{\xi}, z) + \zeta b(z, u) + \zeta b(u, \check{\varrho})]$ for all $\check{\xi}, \check{\varrho}, z, u \in \mathcal{Z}$ for some $s \geq 1$ (rectangle inequality).*

The pair $(\mathcal{Z}, \zeta b)$ is called a "rectangular b-metric space."

Very recently, Mustafa, Z. *et. al.* 2019 [33] further extended rectangular b-metric space and has proved an analogue result of the contraction principle. Karapinar, 2013 [36] attempted to propose the idea of a quasi partial metric by removing the condition of symmetry from a partial metric.

Definition 6. *(Karapinar [36]): Let $\mathcal{Z}$ be a non empty set. A function $qp: \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^+$ is defined to be a "quasi-partial metric on $\mathcal{Z}$" if the conditions specified below hold :*

1. *$qp(\check{\xi}, \check{\xi}) = qp(\check{\xi}, \check{\varrho}) = qp(\check{\varrho}, \check{\varrho})$ implies $\check{\xi} = \check{\varrho}$,*

2. *$qp(\check{\xi}, \check{\xi}) \leq qp(\check{\xi}, \check{\varrho})$,*

3. $qp(\check{\xi}, \check{\xi}) \leq qp(\check{\varrho}, \check{\xi})$,

4. $qp(\check{\xi}, \check{\varrho}) \leq qp(\check{\xi}, z) + qp(z, \check{\varrho}) - qp(z, z)$ *for all* $\check{\xi}, \check{\varrho}, z \in \mathcal{Z}$.

The pair $(\mathcal{Z}, qp)$ *is called a "quasi-partial metric space."*

Gupta and Gautam, 2015 [37] further merged the concepts of quasi partial metric and b-metric to construct a quasi partial b-metric. The authors proved various fixed point results in this space.

Definition 7. *(Gupta and Gautam [37]): A function* $qp_b: \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^+$ *is defined to be a "quasi-partial b-metric on* $\mathcal{Z}$*" if for some real number* $s \geq 1$ *and all* $\check{\xi}, \check{\varrho}, z \in \mathcal{Z}$:

1. $qp_b(\check{\xi}, \check{\xi}) = qp_b(\check{\xi}, \check{\varrho}) = qp_b(\check{\varrho}, \check{\varrho})$ *implies* $\check{\xi} = \check{\varrho}$,

2. $qp_b(\check{\xi}, \check{\xi}) \leq qp_b(\check{\xi}, \check{\varrho})$,

3. $qp_b(\check{\xi}, \check{\xi}) \leq qp_b(\check{\varrho}, \check{\xi})$,

4. $qp_b(\check{\xi}, \check{\varrho}) \leq s[qp_b(\check{\xi}, z) + qp_b(z, \check{\varrho})] - qp_b(z, z)$ *for all* $\check{\xi}, \check{\varrho}, z \in \mathcal{Z}$.

The pair $(\mathcal{Z}, qp_b)$ *is called a "quasi partial b-metric space."*

Jachymski, 2007 [20] generalized the notion of a partially ordered metric space by introducing the concept of a metric space endowed with a graph G. He introduced a special extended class of contraction mapping called G-contractions and successfully extended fixed point results concerning such mappings in such a space.

Definition 8. *(Jachymski [20]): Let* $(\mathcal{Z}, \zeta)$ *be a metric space equipped with graph* G. *The mapping* $\mho: \mathcal{Z} \to \mathcal{Z}$ *is defined to be a "*G*-contraction mapping" if the conditions specified below hold :*

1. $\mho$ *is edge preserving, that is :* $(\check{\xi}, \check{\varrho}) \in E_{\mathrm{G}} \Rightarrow (\mho\check{\xi}, \mho\check{\varrho}) \in E_{\mathrm{G}}$.

2. $\exists k \in [0,1)$ *such that* $\zeta(\mho\check{\xi}, \mho\check{\varrho}) \leq k\zeta(\check{\xi}, \check{\varrho})$ *for all* $(\check{\xi}, \check{\varrho}) \in E_{\mathrm{G}}$.

Bojor, F., 2012 [23] using the idea of G-contraction, established the notion of G-Kannan mapping and explored certain conditions on the graph G and the mapping to possess fixed points in a metric space.

Definition 9. *(Bojor, F. [23]): Let* $(\mathcal{Z},\zeta)$ *be a metric space equipped with graph* G. *The mapping* $\mathfrak{V}:\mathcal{Z}\to\mathcal{Z}$ *is defined to be a "*G*-Kannan mapping" if the conditions specified below hold:*

1. $\mathfrak{V}$ *is edge preserving, that is :* $(\check{\xi},\check{\varrho})\in E_{\mathrm{G}}\Rightarrow(\mathfrak{V}\check{\xi},\mathfrak{V}\check{\varrho})\in E_{\mathrm{G}}$.

2. $\exists k\in[0,\frac{1}{2})$ *such that* $\zeta(\mathfrak{V}\check{\xi},\mathfrak{V}\check{\varrho})\leq k[\zeta(\check{\xi},\mathfrak{V}\check{\xi})+\zeta(\check{\varrho},\mathfrak{V}\check{\varrho})]$ *for all* $(\check{\xi},\check{\varrho})\in E_{\mathrm{G}}$.

Bojor, F., 2012 [25] further studied the fixed point existence results for Ciric- Reich-Rus operators in framework of metric spaces endowed with a graph G. These mappings were referred to as "Reich G-Contraction" or a "G-Ciric-Reich-Rus operator" thereby generalizing G-Contractions and G-Kannan mappings.

Definition 10. *(Bojor, F. [25]): Let* $(\mathcal{Z},\zeta)$ *be a metric space equipped with graph* G. *The mapping* $\mathfrak{V}:\mathcal{Z}\to\mathcal{Z}$ *is said to be a "*G*-Ciric-Reich-Rus operator" if below mentioned conditions are satisfied:*

1. $\mathfrak{V}$ *is edge preserving, that is :* $(\check{\xi},\check{\varrho})\in E_{\mathrm{G}}\Rightarrow(\mathfrak{V}\check{\xi},\mathfrak{V}\check{\varrho})\in E_{\mathrm{G}}$.

2. $\exists f,g,h$ *such that* $f+g+h<1$ *and satisfying* $\zeta(\mathfrak{V}\check{\xi},\mathfrak{V}\check{\varrho})\leq f\zeta(\check{\xi},\check{\varrho})+g\zeta(\check{\xi},\mathfrak{V}\check{\xi})+h\zeta(\check{\varrho},\mathfrak{V}\check{\varrho})$ *for all* $(\check{\xi},\check{\varrho})\in E_{\mathrm{G}}$.

On similar lines, Fallahi, K., 2016 [24] discussed G-Chatterjea mappings in metric spaces with graph as below.

Definition 11. *(Fallahi, K. [24]): Let* $(\mathcal{Z},\zeta)$ *be a metric space equipped with graph* G. *The mapping* $\mathfrak{V}:\mathcal{Z}\to\mathcal{Z}$ *is defined to be a "*G*-Chatterjea mapping" if the conditions specified below hold :*

1. $\mathfrak{V}$ *is edge preserving, that is :* $(\check{\xi},\check{\varrho})\in E_{\mathrm{G}}\Rightarrow(\mathfrak{V}\check{\xi},\mathfrak{V}\check{\varrho})\in E_{\mathrm{G}}$.

2. $\exists k\in[0,\frac{1}{2})$ *such that* $\zeta(\mathfrak{V}\check{\xi},\mathfrak{V}\check{\varrho})\leq k[\zeta(\check{\xi},\mathfrak{V}\check{\varrho})+\zeta(\check{\varrho},\mathfrak{V}\check{\xi})]$ *for all* $(\check{\xi},\check{\varrho})\in E_{\mathrm{G}}$.

For a fundamental comparison and development of various contractive conditions, one may refer Rhoades [7]-[10], Collaco and Silva [11], Murthy [12], Singh and Tomar [13] and Pant *et. al.* [14].

Thus a huge development is reported in the study of fixed point theory of self-maps in different directions with different approaches. This in turn enhances the applicability of various fixed point results to diverse disciplines of mathematics, statistics, engineering and economics. In the present work, our attempt is to extend, improve and generalize several recent results and explore them for application purpose.

3. Objectives of Research

As discussed in review of literature, several researchers have constructed some new contraction mappings and have contributed to various fixed point results for such mappings. Many other scholars have modified the existing mappings or space and have proved fixed point existence results for these mappings.

- If a fixed point result is stated and proved for a self-mapping $\mathcal{T}$ (with a certain set of conditions say $\mathcal{J}$) defined on a set $\mathcal{Z}$ which is equipped with some structure $\mathcal{S}$, then it is aimed to explore the following problems in this research work so as to extend the result:
    a. Is it possible to weaken $\mathcal{J}$?
    b. Is it possible to alter $\mathcal{S}$?
    c. With a suitable alteration in $\mathcal{S}$, is it possible to weaken $\mathcal{J}$?
    d. With a suitable alteration in $\mathcal{S}$, is it possible to strengthen $\mathcal{J}$?
- Further, some applications of the results in solving certain related mathematical problems would be explored in this work.

4. Methodology to be Adopted

Fixed point theory broadly consists of results involving fixed point properties of mappings often inspired by the structure of the underlying set/space or the properties of the involved mapping.

The research methodology that would be followed comprise of the following steps:

- Investigating the set of conditions on the ambient set/space.

- Developing and investigating properties of the underlying mapping in order to ensure the existence of fixed points under the framework of underlying space.
- Verifying the uniqueness of fixed point.

5. Proposed/Expected Outcome of Research

Various generalizations of fixed point theorems involves either a change in the type of contractions or a change in the type of metric space, usually a more general metric space is considered. These generalizations become an important tool to cover a larger class of functions or spaces, hence more is its scope of application fields. Moreover, the existing results would be subsumed in our results thereby proving the effectiveness of our results.

Our results could serve in future as an answer to resolve various existential problems emanating from differential equations, integral equations, system of linear equations, and several others. Thus, the significance of this work is that it may provide a unified approach in solving equations which may not necessarily be linear in nature. In fact, this area of fixed point theory interests many scholars because of the awareness of its potential for applications.

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Abstract

The short synopsis consists of **five sections** namely, Introduction, Literature review, Objectives of research, Methodology to be adopted and Proposed/Expected Outcome of Research.

The **first section** sets a brief introduction to the evolution and development of fixed point theory and presents some concerned basic fundamentals.

The **second section**is divided into **two subsections**. The **first subsection**, titled Generalizations of Contractive Conditions in Metric Spaces aims to highlight some extentions of Banach Contraction Principle within the framework of metric spaces. On the other hand, **the second subsection** titled Extensions of Fixed Point Results in Various Spaces focuses on several definitions and results related to extension of several contraction mappings in the setting of some more general spaces.

The **third section** deals with various objectives of the underlying research.

The **fourthsection** seeks to explore the methodology involved in underlying research.

The **fifth section** discusses some proposed and expected outcomes of the underlying research.

Keywords: fixed point, banach contraction, kannan contraction, G-contraction, F-contraction.

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